Hawkes 跳扩散模型下的脆弱期权定价

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摘要: 研究了随机利率下含交易对手风险的期权定价问题. 假设具有随机波动率的标的资产价格与交易对手资产价值是随机相关的, 且假设它们的跳跃都服从具有自刺激性的 Hawkes 过程. 针对所构建的期权定价模型, 求得了脆弱欧式期权价格的半解析表达式. 数值分析中, 通过快速傅里叶变换方法计算期权价格, 发现所构建模型的期权价格要高于 Poisson 跳模型、固定相关模型和固定利率模型. 此外, 期权价值随违约边界值和破产成本比例的增大而减小; 期权价值随标的初始价格和交易对手资产初始价值的增大而增大.

关键词: 期权定价; Hawkes 过程; 随机相关; 随机利率

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Vulnerable option pricing under the Hawkes jump-diffusion model

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Abstract: This paper investigates the issue of option pricing with counterparty risk under stochastic interest rate. Assume that the value of the underlying asset with stochastic volatility and the value of the counterparty asset are randomly correlated, and assume that their jumps are driven by the self-exciting Hawkes processes. The semi-analytical expression for the vulnerable European option price under the proposed model is obtained. In the numerical analysis, the option price is calculated by the fast Fourier transform method, and find the option price under the constructed model is higher than the Poisson jump model, the constant correlation model and the constant interest rate model. Moreover, option price decreases with the default boundary and the bankruptcy cost ratio; option price increases with the underlying initial price and the counterparty asset initial value.

Key words: option pricing; Hawkes processes; stochastic correlation; stochastic interest rate

1 引　言

近年来, 随着市场不确定性上升, 市场参与者管理个性化风险的需求增强, 从而使得场外期权交易规模不断扩大. 场外期权是非集中场所交易的非标准化期权合约, 其往往面临着交易对手违约风险, 所以许多学者也把它称为脆弱期权. 自 2007 年∼2008 年金融危机爆发以来, 许多大型金融机构和公司都面临破产倒闭风险, 投资者对于场外期权遭受信用违约风险的担忧也日益增加, 因此在场外期权定价时考虑信用风险是非常必要的.

Johnson 等[1]最先在 Merton[2]的公司债券违约模型上研究含有信用风险的期权定价问题. 在标的资产与交易对手资产相互独立的前提下, Hull 等[3]构建了含有信用风险的衍生证券约化定价模型; 相关的研究还

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包括文献[4, 5]. Klein[6]对文献[3]进行了拓展, 允许标的资产与交易对手风险是相关性的, 同时考虑期权发行人有其他负债. Klein 等[7] 研究了违约边界取决于期权和交易对手债务价值的脆弱期权定价问题. 吴恒煜等[8]在文献[6]的基础上研究了不完全市场下有违约风险的欧式期权定价. Liao 等[9]考虑了交易对手在期权到期之前可能发生的违约, 而且还考虑了交易对手的总资产, 标的股票和无违约零息债券之间的相关性并得到解析定价公式. Liang 等[10]结合实际违约的触发机制探讨了场外交易市场欧式期权的信用风险和估值模型.

然而, 上述文献关于脆弱期权定价模型的研究不尽完美的善. 例如, 常波动率假设与实际市场中所观察到的波动率聚集、尖峰厚尾和波动率均值回复性等相矛盾, 常利率假设与实际市场利率随时变化不一致. 于是, 有学者在脆弱期权定价模型中考虑随机波动率[11,12]和随机利率[13,14]. 此外, 脆弱期权定价的大部分文献都假设资产价值遵循对数正态扩散过程. 该假设忽略了新信息的到来和突发性灾难事件的发生可能导致资产价格的不连续变化. 为了克服该缺陷, 有学者通过假设标的资产和交易对手资产都遵循泊松跳扩散过程来刻画资产价格的不连续变化[15,16]. 泊松跳过程具有独立增量性, 其刻画的跳跃到达时间间隔相互独立. 但学者们近年来发现跳跃并非独立到达, 而是广泛存在聚集现象[17−20]. 于是, 一些学者开始采用能刻画跳跃聚集特征的点过程如Hawkes 过程对资产价格跳跃进行建模[21,22], 他们发现能刻画跳跃聚集的期权定价模型都能显著改善定价能力. Ma 等[23] 提出一个基于 Hawkes 跳扩散的脆弱期权定价模型, 但该模型假设利率和波动率都是常数.

另一方面, 在金融衍生品定价和风险管理中, 资产之间的相关性起着至关重要的作用. 简单地使用确定的相关性往往容易导致定价和风险度量偏误, 这是因为真实市场中的相关性并确定[24]. 之后, 有学者在对外国股票期权和双币种期权定价时考虑了随机相关[25−27]. 然而, 现有的脆弱期权定价模型尚未考虑随机相关.

考虑到随机利率、随机波动率、随机相关和跳跃集聚的现实重要性, 本文将在 Ma 等[23] 的模型基础上进行拓展, 构建更符合市场实际的脆弱期权定价模型. 具体而言, 首先, 将标的资产与交易对手资产之间的随机相关性纳入考虑, 并采用 Ornstein-Uhlenbeck(OU)过程对其建模; 其次, 将 Hull-White 随机利率纳入模型; 最后, 纳入由 Cox-Ingerson-Ross(CIR)过程驱动的标的资产波动率. 针对所构建的脆弱期权定价模型, 通过傅里叶变换方法求得欧式脆弱期权价值的半解析表达式.

2 Hawkes 跳扩散模型

令 $(\Omega,\mathcal{F},(\mathcal{F}(t))_{0\leqslant t\leqslant T},Q)$为完备概率空间, 其中 T 为期权到期时间, $(\mathcal{F}(t))_{0\leqslant t\leqslant T}$ 表示信息流, Q 是风险中性测度, E 表示 Q 下的期望. 令 $S(t)$和 $V(t)$分别表示标的资产和交易对手资产在 t 时刻的价值, 假设 $S(t)$ 和 $V(t)$在概率空间$(\Omega,\mathcal{F},Q)$中分别满足随机微分方程

$$\frac{\mathrm{d}S(t)}{S(t-)}=(r(t)-\lambda_1(t)m_1)\mathrm{d}t+\sqrt{\nu(t)}\mathrm{d}W^{\mathrm{s}}(t)+(\mathrm{e}^{\xi_1(t)}-1)\mathrm{d}N_1(t), \tag{1}$$

$$\frac{\mathrm{d}V(t)}{V(t-)}=(r(t)-\lambda_2(t)m_2)\mathrm{d}t+\sigma\mathrm{d}W(t)+(\mathrm{e}^{\xi_2(t)}-1)\mathrm{d}N_2(t), \tag{2}$$

其中 $\sigma>0$ 是常数.

随机利率 $r(t)$和瞬时方差 $\nu(t)$满足如下随机微分方程

$$\mathrm{d}r(t)=\beta_{\mathrm{r}}(\theta_{\mathrm{r}}-r(t))\mathrm{d}t+\sigma_{\mathrm{r}}\mathrm{d}W^{\mathrm{r}}(t), \tag{3}$$

$$\mathrm{d}\nu(t)=\beta_\nu(\theta_\nu-\nu(t))\mathrm{d}t+\sigma_\nu\sqrt{\nu(t)}\mathrm{d}W^\nu(t). \tag{4}$$

对每个 $j\in\{1,2\}$, 假设跳跃幅度 $\xi_j(t)$是独立同分布的随机变量序列且 $\xi_j(t)\sim N(\mu_j,\sigma_j^2)$, 则 $m_j=\mathrm{E}\left[\mathrm{e}^{\xi_j(t)}-1\right]$. $N_j(t)$是强度为 $\lambda_j(t)$的 Hawkes 过程, 其中

$$\mathrm{d}\lambda_j(t)=\eta_j(\lambda_{j,\infty}-\lambda_j(t))\mathrm{d}t+\gamma_{jj}\mathrm{d}N_j(t), \tag{5}$$

$W^{\mathrm{s}}(t)$, $W^{\nu}(t)$, $W(t)$, $W^{\mathrm{r}}(t)$都是测度 Q 下的标准布朗运动. $\mathrm{d}W^{\mathrm{s}}(t)\mathrm{d}W^{\nu}(t) = \rho_1\mathrm{d}t$, ρ_1 是一常数, 而且 $\mathrm{d}W^{\mathrm{s}}(t)\mathrm{d}W(t) = \rho(t)\mathrm{d}t$, 其中 $\rho(t)$是满足以下方程的 OU 过程

$$\mathrm{d}\rho(t) = \kappa_\rho\left(\theta_\rho - \rho(t)\right)\mathrm{d}t + \eta_\rho \mathrm{d}W^{\rho}(t), \tag{6}$$

$W^{\rho}(t)$是测度 Q 下的标准布朗运动, 且 $W^{\mathrm{r}}(t)$ 和 $W^{\rho}(t)$都独立于其它任何布朗运动.

根据文献[28], $\frac{\sqrt{\kappa_\rho}}{\eta_\rho} \geqslant 3$ 是 $-1 < \rho(t) < 1$ 的充分条件.

令 $X(t) = \ln S(t), Y(t) = \ln\, V(t)$. 根据伊藤公式, 有

$$\mathrm{d}X(t) = \left(r(t) - \lambda_1(t)m_1 - \frac{1}{2}\nu(t)\right)\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W^{\mathrm{s}}(t) + \mathrm{d}J_1(t), \tag{7}$$

$$\mathrm{d}Y(t) = \left(r(t) - \lambda_2(t)m_2 - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}W(t) + \mathrm{d}J_2(t), \tag{8}$$

其中 $\mathrm{d}J_j(t) = \xi_j(t)\mathrm{d}N_j(t), j = 1, 2$.

到期时间为 T 的零息债券在时间 $0 \leqslant t \leqslant T$ 的价格 $P(t,T)$可表示为 $P(t,T) = \mathrm{E}_t\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\right]$.

若无风险利率 $r(t)$由 Hull-White 利率模型决定, 即其满足式(3), 则

$$P(t,T) = \exp(M(t,T) - L(t,T)r(t)), \tag{9}$$

其中 $M(t,T) = (\theta_{\mathrm{r}} - \frac{\sigma_{\mathrm{r}}^2}{2\beta_{\mathrm{r}}^2})(L(t,T) - (T-t)) - \frac{\sigma_{\mathrm{r}}^2}{4\beta_{\mathrm{r}}}(L(t,T))^2, L(t,T) = \frac{1}{\beta_{\mathrm{r}}}(1 - \mathrm{e}^{-\beta_{\mathrm{r}}(T-t)})$.

通过以下 Radon-Nikodym 导数定义前向测度 Q^{T}, 即

$$\frac{\mathrm{d}Q^{\mathrm{T}}}{\mathrm{d}Q} = \frac{\mathrm{e}^{-\int_0^T r(u)\mathrm{d}u}}{P(0,T)}. \tag{10}$$

在测度 Q^{T}下, 随机利率 $r(t)$满足随机微分方程

$$\mathrm{d}r(t) = (\beta_{\mathrm{r}}(\theta_{\mathrm{r}} - r(t)) - \sigma_{\mathrm{r}}^2 L(t,T))\mathrm{d}t + \sigma_{\mathrm{r}}\mathrm{d}W_{\mathrm{r}}^{\mathrm{T}}(t), \tag{11}$$

其中满足 $\mathrm{d}W_{\mathrm{r}}^{\mathrm{T}}(t) = \mathrm{d}W^{\mathrm{r}}(t) + \sigma_{\mathrm{r}}L(t,T)\mathrm{d}t$ 的 $W_{\mathrm{r}}^{\mathrm{T}}(t)$ 在测度 Q^{T} 下是标准布朗运动.

3　特征函数

本节将求得 $X(T)$和 $Y(T)$在测度 Q^{T} 下的联合条件特征函数. 令 $\boldsymbol{Z}(t) = (X(t), Y(t))^{\mathrm{T}}$, $\boldsymbol{\phi}=(\phi_1, \phi_2)^{\mathrm{T}}$, 则 $\boldsymbol{Z}(T)$的条件特征函数为

$$f(\phi_1, \phi_2, \tau) = \mathrm{E}^{Q^{\mathrm{T}}}\left[\mathrm{e}^{i\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{Z}(T)} \mid \mathcal{F}(t)\right], \tag{12}$$

其中 $i = \sqrt{-1}, \tau = T - t$.

令 $\boldsymbol{Z}^{\mathrm{J}}(t)$和 $\boldsymbol{Z}^{\mathrm{C}}(t)$满足随机微分方程

$$\mathrm{d}\boldsymbol{Z}^{\mathrm{J}}(t) = (-\lambda_1(t)m_1\mathrm{d}t + \mathrm{d}J_1(t), -\lambda_2(t)m_2\mathrm{d}t + \mathrm{d}J_2(t))^{\mathrm{T}},$$

$$\mathrm{d}\boldsymbol{Z}^{\mathrm{C}}(t) = (\mathrm{d}X^{\mathrm{C}}(t), \mathrm{d}Y^{\mathrm{C}}(t))^{\mathrm{T}},$$

其中

$$\mathrm{d}X^{\mathrm{C}}(t) = (r(t) - \frac{1}{2}\nu(t))\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W^{\mathrm{s}}(t),$$

$$\mathrm{d}Y^{\mathrm{C}}(t) = (r(t) - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma\mathrm{d}W(t).$$

则显然有 $\boldsymbol{Z}(t) = \boldsymbol{Z}^{\mathrm{J}}(t) + \boldsymbol{Z}^{\mathrm{C}}(t)$. 因为 $J_j(t), j = 1, 2$ 与 $W^{\mathrm{s}}(t), W(t), W^{\nu}(t), W_{\mathrm{r}}^{\mathrm{T}}(t), W^{\rho}(t)$相互独立, 所以

$$f(\phi_1, \phi_2, \tau) = f_{\mathrm{C}}(\phi_1, \phi_2, \tau)f_{\mathrm{J}}(\phi_1, \phi_2, \tau), \tag{13}$$

其中 $f_{\mathrm{C}}(\phi_1,\phi_2,\tau)=\mathrm{E}^{Q\mathrm{T}}\left[\mathrm{e}^{i\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{C}}(T)}\mid\mathcal{F}(t)\right]$, $f_{\mathrm{J}}(\phi_1,\phi_2,\tau)=\mathrm{E}^{Q\mathrm{T}}\left[\mathrm{e}^{i\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{J}}(T)}\mid\mathcal{F}(t)\right]$.

首先求 $f_{\mathrm{J}}(\phi_1,\phi_2;\tau,X,Y,\rho,\nu,r)$. 令

$$\tilde{\boldsymbol{Y}}(t)=(\lambda_1(t),\ \lambda_2(t),\ -m_1\int_0^t\lambda_1(s)\mathrm{d}s+J_1(t),\ -m_2\int_0^t\lambda_2(s)\mathrm{d}s+J_2(t))^{\mathrm{T}},$$

则 $\tilde{\boldsymbol{Y}}(t)$是一个满足以下方程的四维仿射跳扩散过程

$$\begin{aligned}&\mathrm{d}\tilde{\boldsymbol{Y}}(t)=(\tilde{\boldsymbol{\alpha}}+\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{Y}}(t))\mathrm{d}t+\sum_{j=1}^{2}\boldsymbol{\gamma}_j\mathrm{d}\boldsymbol{K}_j(t),\\&\lambda_j(t)=\tilde{\boldsymbol{\delta}}_j\tilde{\boldsymbol{Y}}(t),\end{aligned}$$

其中 $\tilde{\boldsymbol{\alpha}}=(\eta_1\lambda_{1,\infty},\eta_2\lambda_{2,\infty},\mathbf{0}_{\mathrm{V}})^{\mathrm{T}}$, $\tilde{\boldsymbol{\beta}}=\begin{pmatrix}\mathrm{diag}(-\eta_1,-\eta_2)&\mathbf{0}_{\mathrm{M}}\\\mathrm{diag}(-m_1,-m_2)&\mathbf{0}_{\mathrm{M}}\end{pmatrix}$, $\boldsymbol{K}_j(t)=(N_j(t)\mathbf{1}_{\mathrm{V}},J_j(t)\mathbf{1}_{\mathrm{V}})^{\mathrm{T}}$, $\boldsymbol{\gamma}_1=\mathrm{diag}\left(\gamma_{11},0,\boldsymbol{e}_1\right)^{\mathrm{T}}$, $\boldsymbol{\gamma}_2=\mathrm{diag}\left(0,\gamma_{22},\boldsymbol{e}_2\right)^{\mathrm{T}}$, $\tilde{\boldsymbol{\delta}}_j=(\boldsymbol{e}_j,\mathbf{0}_{\mathrm{V}})$. $\mathbf{0}_{\mathrm{V}}$ 表示元素全为 0 的二维行向量, $\mathbf{0}_{\mathrm{M}}$ 表示元素全为 0 的 2×2 矩阵, $\mathbf{1}_{\mathrm{V}}$ 表示元素全为 1 的二维行向量. $\boldsymbol{e}_j$ 表示二维行向量中第 j 个元素为 1, 其它元素为 0.

根据文献[29], $\tilde{Y}(T)$在时间 t 的条件特征函数为

$$f_{\tilde{Y}}(\boldsymbol{\varepsilon},\tau)=\mathrm{E}^{Q\mathrm{T}}\left[\mathrm{e}^{\boldsymbol{\varepsilon}^{\mathrm{T}}\tilde{\boldsymbol{Y}}(T)}\mid\mathcal{F}(t)\right]=\exp\left(G_1(\boldsymbol{\varepsilon},\tau)+\boldsymbol{G}_2(\boldsymbol{\varepsilon},\tau)^{\mathrm{T}}\tilde{\boldsymbol{Y}}(t)\right),\quad\boldsymbol{\varepsilon}\in\mathbb{C}^4,$$

$\boldsymbol{G}_2(\boldsymbol{\varepsilon},\tau)=(G_{21}(\boldsymbol{\varepsilon},\tau),G_{22}(\boldsymbol{\varepsilon},\tau),G_{23}(\boldsymbol{\varepsilon},\tau),G_{24}(\boldsymbol{\varepsilon},\tau))^{\mathrm{T}}$满足以下微分方程

$$\begin{cases}\dfrac{\partial\boldsymbol{G}_1(\boldsymbol{\varepsilon},\tau)}{\partial t}=-\tilde{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{G}_2(\boldsymbol{\varepsilon},\tau)\\\dfrac{\partial\boldsymbol{G}_2(\boldsymbol{\varepsilon},\tau)}{\partial t}=-\tilde{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{G}_2(\boldsymbol{\varepsilon},\tau)-\displaystyle\sum_{j=1}^{2}(\Psi_j(\gamma_j\boldsymbol{G}_2(\boldsymbol{\varepsilon},\tau))-1)\tilde{\boldsymbol{\delta}}_j^{\mathrm{T}},\end{cases}\tag{14}$$

其中 $\Psi_j(\boldsymbol{\omega})=\int_{\mathbb{R}}\mathrm{e}^{\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\varphi}(y)}\mathrm{d}F_j(y),\boldsymbol{\omega}\in\mathbb{C}^4,\boldsymbol{\varphi}(y)=(\mathbf{1}_{\mathrm{V}},y\mathbf{1}_{\mathrm{V}})^{\mathrm{T}}$, $F_j(y)$ 是均值为 μ_j、方差为 σ_j^2 的正态分布函数.

此外, 边界条件 $G_1(\boldsymbol{\varepsilon},0)=0,\boldsymbol{G}_2(\boldsymbol{\varepsilon},0)=\boldsymbol{\varepsilon}$. 从而可得

$$f_{\mathrm{J}}(\phi_1,\phi_2,\tau)=f_{\tilde{Y}}((\mathbf{0}_{\mathrm{V}},i\phi_1,i\phi_2),\tau).\tag{15}$$

方程组(14)通常没有闭形式解, 将采用 Runge-Kutta 算法对其进行求解, 进而求出式(15).

下面求解 $f_{\mathrm{C}}(\phi_1,\phi_2,\tau)$. 在测度 Q^{T}下, 标的资产连续部分等价于

$$\mathrm{d}\boldsymbol{Z}^{\mathrm{C}}(t)=\left((r(t)-\frac{1}{2}\nu(t))\mathrm{d}t+\sqrt{\nu(t)}\mathrm{d}\tilde{W}^{\mathrm{s}}(t),\ (r(t)-\frac{1}{2}\sigma^2)\mathrm{d}t+\sigma(\rho(t)\mathrm{d}\tilde{W}^{\mathrm{s}}(t)+\sqrt{1-\rho^2(t)}\mathrm{d}\tilde{W}(t))\right)^{\mathrm{T}},\tag{16}$$

其中 $\mathrm{d}\nu(t)=\beta_\nu(\theta_\nu-\nu(t))\mathrm{d}t+\sigma_\nu\sqrt{\nu(t)}(\rho_1\mathrm{d}\tilde{W}^{\mathrm{s}}(t)+\sqrt{1-\rho_1^2}\mathrm{d}\tilde{W}^{\nu}(t)),\mathrm{d}r(t)=(\beta_{\mathrm{r}}(\theta_{\mathrm{r}}-r(t))-\sigma_{\mathrm{r}}^2L(t,T))\mathrm{d}t+\sigma_{\mathrm{r}}\mathrm{d}W_{\mathrm{r}}^{\mathrm{T}}(t),\mathrm{d}\rho(t)=\kappa_\rho\left(\theta_\rho-\rho(t)\right)\mathrm{d}t+\eta_\rho\mathrm{d}W^{\rho}(t)$. $\tilde{W}^{\mathrm{s}}(t),\tilde{W}(t),\tilde{W}^{\nu}(t)$是三个相互独立的标准布朗运动.

有下列结论.

定理 1 如果 $\boldsymbol{Z}^{\mathrm{C}}(t)$由式(16)确定, 则 $X^{\mathrm{C}}(t)$和式$Y^{\mathrm{C}}(t)$的联合特征函数为

$$f_{\mathrm{C}}(\phi_1,\phi_2,\tau)=\exp\left(A(\phi_1,\phi_2,\tau)\nu+B(\phi_1,\phi_2,\tau)r+C(\phi_1,\phi_2,\tau)\rho+D(\phi_1,\phi_2,\tau)+i\phi_1X^{\mathrm{C}}+i\phi_2Y^{\mathrm{C}}\right),$$

其中

$$\begin{aligned}&A(\phi_1,\phi_2,\tau)=\tilde{d}(\phi_1)+d(\phi_1)\sigma_\nu^2\frac{1-\mathrm{e}^{-d(\phi_1)\tau}}{1-g(\phi_1)\mathrm{e}^{-d(\phi_1)\tau}},\\&B(\phi_1,\phi_2,\tau)=\frac{i(\phi_1+\phi_2)(1-\mathrm{e}^{-\beta_{\mathrm{r}}\tau})}{\beta_{\mathrm{r}}},\\&C(\phi_1,\phi_2,\tau)=\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)m}{\kappa_\rho}+\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)n}{l+\kappa_\rho}\mathrm{e}^{l(\tau-T)}-\end{aligned}$$

$$\frac{\rho_1\sigma\sigma_\nu i\phi_2 C_1 m}{\kappa_\rho - l_1}\mathrm{e}^{-l_1\tau} - \frac{\rho_1\sigma\sigma_\nu i\phi_2 C_1 n}{\kappa_\rho + l - l_1}\mathrm{e}^{-lT+(l-l_1)\tau} + C_2\mathrm{e}^{-\kappa_\rho\tau},$$

$$\begin{aligned}
D(\phi_1,\phi_2,\tau) = {} & \beta_\nu\theta_\nu H_0(\phi_1,\tau) + \kappa_\rho\theta_\rho H_1(\phi_1,\phi_2,\tau) + \beta_{\mathrm{r}}\theta_{\mathrm{r}} H_2(\phi_1,\phi_2,\tau) - \sigma_{\mathrm{r}}^2 H_3(\phi_1,\phi_2,\tau) + \\
& \frac{1}{2}\sigma_{\mathrm{r}}^2 H_4(\phi_1,\phi_2,\tau) + \frac{1}{2}\eta_\rho^2 H_5(\phi_1,\phi_2,\tau) - \frac{1}{2}\sigma^2\phi_2(\phi_2+i)\tau,\\
d(\phi_1) = {} & \sqrt{(\rho_1\sigma_\nu i\phi_1 - \beta_\nu)^2 + \sigma_\nu^2(i\phi_1+\phi_1^2)},\\
g(\phi_1) = {} & \frac{\tilde{d}(\phi_1) + d(\phi_1)}{\tilde{d}(\phi_1) - d(\phi_1)},\\
C_2 = {} & -\frac{(\rho_1\sigma\sigma_\nu i\phi_2 C_1 - \sigma\phi_1\phi_2)m}{\kappa_\rho} - \frac{(\rho_1\sigma\sigma_\nu i\phi_2 C_1 - \sigma\phi_1\phi_2)n}{l+\kappa_\rho}\mathrm{e}^{-lT} + \\
& \frac{\rho_1\sigma\sigma_\nu i\phi_2 C_1 m}{\kappa_\rho - l_1} + \frac{\rho_1\sigma\sigma_\nu i\phi_2 C_1 n}{\kappa_\rho + l - l_1}\mathrm{e}^{-lT},\\
\tilde{d}(\phi_1) = {} & \rho_1\sigma_\nu i\phi_1 - \beta_\nu,\\
C_1 = {} & (i\phi_1+\phi_1^2)/(\sigma_\nu^2/2 - d(\phi_1)),
\end{aligned}$$

$m, n, l, l_1, H_j(\phi_1,\phi_2,\tau), j = 0, 1, \ldots, 5$ 的具体表达式以及该定理的证明见附录 A.

最终求得 $X(t)$和 $Y(t)$的联合特征函数为

$$f(\phi_1,\phi_2,\tau) = f_{\mathrm{C}}(\phi_1,\phi_2,\tau) f_{\mathrm{J}}(\phi_1,\phi_2,\tau). \tag{17}$$

4 脆弱欧式期权定价

根据风险中性定价原理, 执行价格为 K, 到期日为 T 的脆弱欧式看涨期权在时间 $0 \leqslant t \leqslant T$ 的价值为

$$\begin{aligned}
C(X(t),Y(t),\rho(t),\nu(t),r(t)) = {} & \mathrm{E}^Q\Big[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\Big(1_{\{V(T)\geqslant D^*\}}(S(T)-K)^+ + \\
& 1_{\{V(T)<D^*\}}\frac{(1-\omega)V(T)(S(T)-K)^+}{D}\Big)\Big|\,\mathcal{F}(t)\Big],
\end{aligned}\tag{18}$$

其中 ω 是破产成本比例. D 是交易对手总负债, 常数违约边界 D^* 小于 D, 这是因为交易对手资不抵债时还可能继续保持经营. 当信用事件发生时, 期权卖方支付比例为 $\frac{(1-\omega)V(T)}{D}$, 若 $V(T) \geqslant D^*$, 则信用事件不会发生, 期权持有者在 T 时刻将获得全部支付. 通过测度变换式(10), 该期权的价值可表示为

$$\begin{aligned}
C(k,\zeta) := {} & C(X(t),Y(t),\rho(t),\nu(t),r(t)) = P(t,T)\times \\
& \mathrm{E}^{Q^{\mathrm{T}}}\left[1_{\{\mathrm{e}^{Y(T)}\geqslant \mathrm{e}^\zeta\}}\left(\mathrm{e}^{X(T)} - \mathrm{e}^k\right)^+ + 1_{\{\mathrm{e}^{Y(T)}<\mathrm{e}^\zeta\}}\frac{(1-\omega)\mathrm{e}^{Y(T)}(\mathrm{e}^{X(T)}-\mathrm{e}^k)^+}{D}\Big|\,\mathcal{F}(t)\right],
\end{aligned}\tag{19}$$

其中 $X(T) = \ln S(T), Y(T) = \ln V(T), k = \ln K, \zeta = \ln D^*$.

定理 2　若标的物价格 $X(t)$和交易对手资产价值 $Y(t)$的联合特征函数为 $f(\phi_1,\phi_2,\tau)$, 则该脆弱欧式看涨期权的价值为

$$C(k,\zeta) = P(t,T)C_1(k,\zeta) + P(t,T)\frac{(1-\omega)}{D}C_2(k,\zeta), \tag{20}$$

其中

$$C_1(k,\zeta) = \frac{\mathrm{e}^{-\alpha_1 k - \beta_1\zeta}}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i(\delta_1 k + \delta_2\zeta)}\hat{c}_1(\delta_1,\delta_2)\mathrm{d}\delta_1\mathrm{d}\delta_2, \tag{21}$$

$$C_2(k,\zeta) = \frac{\mathrm{e}^{-\alpha_1 k + \beta_2\zeta}}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i(\delta_1 k + \delta_2\zeta)}\hat{c}_2(\delta_1,\delta_2)\mathrm{d}\delta_1\mathrm{d}\delta_2, \tag{22}$$

$$\hat{c}_1(\delta_1,\delta_2)=\frac{f(\delta_1-(\alpha_1+1)i,\delta_2-\beta_1 i,\tau)}{(\alpha_1^2+\alpha_1-\delta_1^2+i(2\alpha_1+1)\delta_1)(\beta_1+i\delta_2)},$$
$$\hat{c}_2(\delta_1,\delta_2)=\frac{f(\delta_1-(\alpha_1+1)i,\delta_2-(1-\beta_2)i,\tau)}{(\alpha_1^2+\alpha_1-\delta_1^2+i(2\alpha_1+1)\delta_1)(\beta_2-i\delta_2)},$$

其中 $\alpha_1,\beta_1,\beta_2>0$ 为阻尼系数. 证明见附录 B.

5 数值分析

这一节, 基于快速傅里叶变换(FFT)方法对构建的期权定价模型进行数值分析. 用于本文期权定价的 FFT 算法过程见附录 C; 在使用 FFT 方法时, 令 $N=2\,048$, $\omega_1=\omega_2=\frac{600}{N}$, $\tilde{\lambda}_1=\tilde{\lambda}_2=\frac{2\pi}{N\omega_1}$, $\alpha_1=1.2$, $\beta_1=\beta_2=2$, $T=1$. 表 1 是基于文献[17, 23, 28, 32]设定的基准参数值. 在模型对比分析中, Poisson 跳模型中的跳强度设定为 Hawkes 跳模型中的长期均值水平(1). 固定相关模型中的常数相关设定为随机相关模型中的长期均值水平(0.5). 固定利率模型中的常利率设定为随机利率模型中的长期均值水平(0.05).

表 1 基准参数值

Table 1 The parameter values in the base case

$S(0)$	$V(0)$	K	$\nu(0)$	$\rho(0)$	$\lambda_1(0)$	$\lambda_2(0)$	$r(0)$	D^*	D	ρ_1
10	10	10	0.02	0.15	1	1	0.02	10	10	−0.5
β_ν	κ_ρ	η_1	η_2	β_{r}	θ_ν	θ_ρ	$\lambda_{1,\infty}$	$\lambda_{2,\infty}$	θ_{r}	σ_ν
1.8	3.5	18	18	0.5	0.02	0.5	1	1	0.05	0.6
η_ρ	γ_{11}	γ_{22}	σ	σ_{r}	μ_1	μ_2	σ_1	σ_2	ω	
0.3	4	4	0.16	0.12	−0.1	−0.1	0.24	0.24	0.5	

图 1 呈现了本文模型与 Poisson 跳模型、固定相关模型和固定利率模型下的(看涨)期权价格与执行价格的关系.

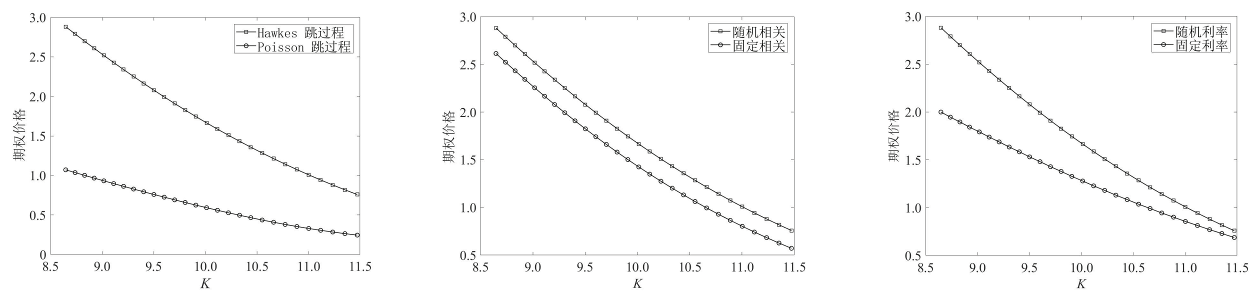

图 1 本文模型与 Poisson 跳模型、固定相关模型和固定利率模型所得脆弱期权价值比较

Fig. 1 The comparison between the vulnerable option prices from the proposed model, Poisson jump model, constant correlation model and constant interest rate model

从图 1 可以看出, 期权价格都随执行价格 K 的增加而逐渐递减, 而且 Hawkes 跳模型的期权价格高于 Poisson 跳模型, 随机相关模型的期权价格高于固定相关模型, 随机利率模型的期权价格高于固定利率, 这是因为随机跳强度模型、随机相关模型和随机利率模型都会导致较大的资产价格不确定性.

图 2 展示了期权价格关于违约边界 D^*、破产成本比例 ω、标的资产初始价格 $S(0)$和交易对手资产初始价值 $V(0)$之间的敏感性分析. 从图中可知, 期权价值随着违约边界值的增加而递减, 这是因为随着违约边界的增加, 交易对手违约的概率会增大; 期权价格随破产成本比例的增大而降低, 这是因为当违约事件发生时, 破产成本比例将会影响回收率, 破产成本比例越大, 回收率越小, 那么造成期权持有人的损失则更大; 期权价格随标的资产初始价格的增大而增大, 这是因为标的资产初始价格越大, 标的资产在到期日超过执行价

格的可能性就越大; 期权价格随交易对手资产初始价值的增大而增大, 这是因为交易对手资产初始价值越大, 其资产的价值在到期日超过违约边界的概率就越大, 从而违约概率越小.

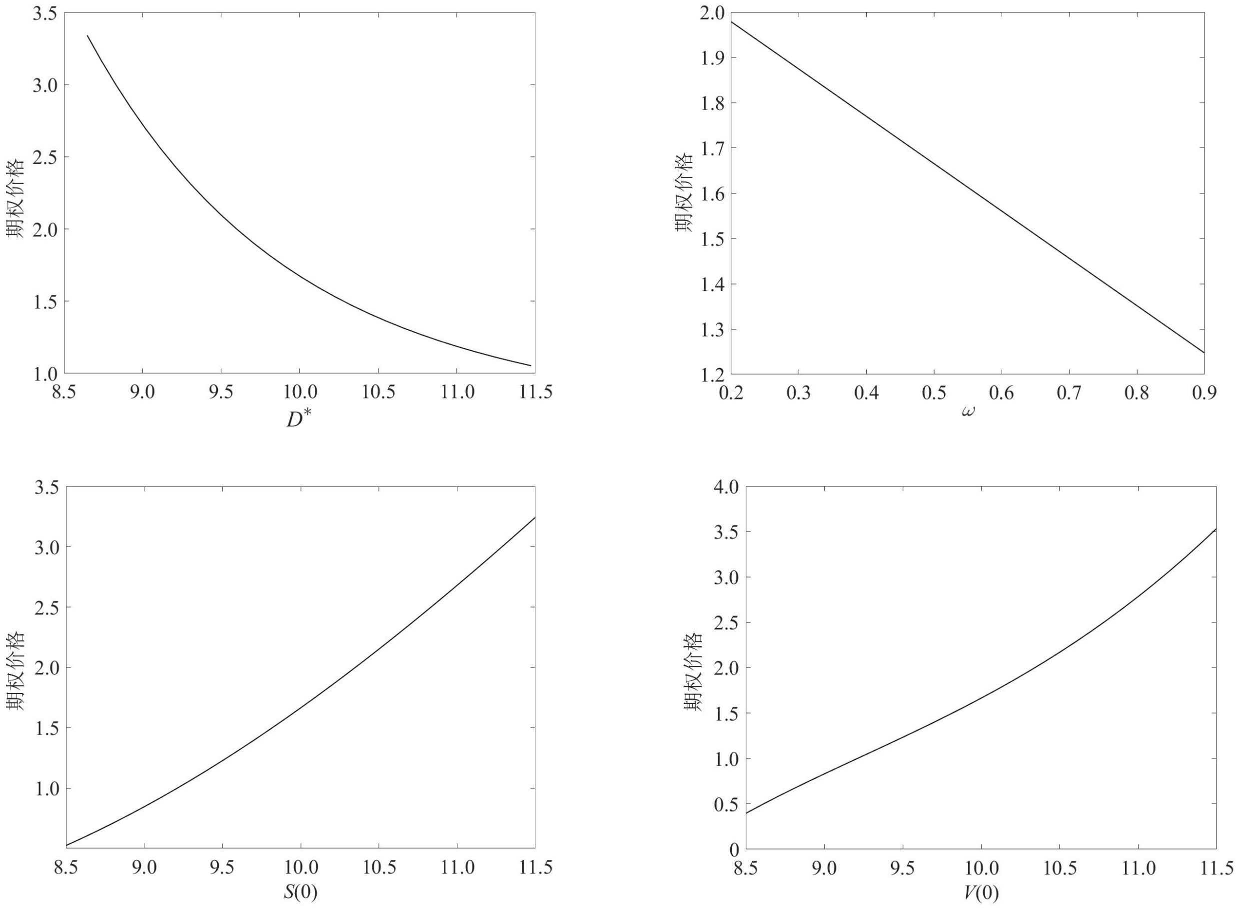

图 2　期权价格与违约边界、破产成本比例、标的资产初始价格和交易对手资产初始价值之间的关系

Fig. 2　The relations between the option prices and default boundary, bankruptcy cost ratio, the underlying price and the counterparty's asset value

6　结束语

本文研究了具有交易对手违约风险的脆弱期权定价问题. 假设标的资产价值与交易对手资产价值都服从自刺激的 Hawkes 跳过程, 且标的资产价值与交易对手资产价值是由 OU 过程驱动的随机相关. 此外, 还将 Hull-White 随机利率纳入模型, 并且考虑标的资产的波动率是由 CIR 过程所驱动. 基于该脆弱期权定价模型, 求得了脆弱欧式(看涨)期权价格的半解析表达式. 通过数值分析, 发现 Hawkes 跳跃的期权价格高于 Poisson 跳跃、具有随机相关性的期权价格高于固定相关性以及具有随机利率的期权价格高于固定利率. 此外, 期权价值随违约边界值的增大而减小; 破产成本越高, 期权价格就越低; 期权价值随标的资产初始价格和交易对手资产初始价值的增加而增加.

参考文献:

[1] Johnson H, Stulz R M. The pricing of options with default risk. Journal of Finance, 1987, 42(2): 267–280.

[2] Merton R C. On the pricing of corporate debt: The risk structure of interest rates. Journal of Finance, 1974, 29(2): 449–470.

[3] Hull J, White A. The impact of default risk on the prices of options and other derivative securities. Journal of Banking and Finance, 1995, 19(2): 299–322.

[4] 林建伟, 李慧敏. 随机利率背景下具有多方担保公司债券的定价. 系统工程学报, 2020, 35(5): 632–641.
Lin J W, Li H M. Pricing of multi-party guarantee corporate bonds in the context of stochastic interest rates. Journal of Systems Engineering, 2020, 35(5): 632–641. (in Chinese)

[5] 梁 进, 包俊利, 曾楚焜. 含信用等级迁移的可违约和可赎回公司债券的结构化定价. 系统工程学报, 2018, 33(6): 793–801.
Liang J, Bao J L, Zeng C K. Pricing on a defaultable and callable corporate bond with credit rating migration under the structure framework. Journal of Systems Engineering, 2018, 33(6): 793–801. (in Chinese)

[6] Klein P. Pricing Black-Scholes options with correlated credit risk. Journal of Banking and Finance, 1996, 20(7): 1211–1229.

[7] Klein P, Inglis M. Pricing vulnerable European options when the option's payoff can increase the risk of financial distress. Journal of Banking and Finance, 2001, 25(5): 993–1012.

[8] 吴恒煜, 吕江林, 肖 峻. 不完全市场下有违约风险的欧式期权定价. 系统工程学报, 2008, 23(4): 405–411.
Wu H Y, Lü J L, Xiao J. Pricing of European option subject to default risk in incomplete markets. Journal of Systems Engineering, 2008, 23(4): 405–411. (in Chinese)

[9] Liao S L, Huang H H. Pricing Black-Scholes options with correlated interest rate risk and credit risk: An extension. Quantitative Finance, 2005, 5(5): 443–457.

[10] Liang G C, Ren X M. The credit risk and pricing of OTC options. Asia-Pacific Financial Markets, 2007, 14(1): 45–68.

[11] Lee M K, Yang S J, Kim J H. A closed form solution for vulnerable options with Heston's stochastic volatility. Chaos, Solitons and Fractals, 2016, 86(5): 23–27.

[12] Wang G Y, Wang X C, Zhou K. Pricing vulnerable options with stochastic volatility. Physica A: Statistical Mechanics and Its Applications, 2017, 485: 91–103.

[13] Hung M W, Liu Y H. Pricing vulnerable options in incomplete markets. Journal of Futures Markets, 2005, 25(2): 135–170.

[14] Ma C Q, Yue S J, Wu H, et al. Pricing vulnerable options with stochastic volatility and stochastic interest rate. Computational Economics, 2020, 56(2): 391–429.

[15] Tian L H, Wang G Y, Wang X C, et al. Pricing vulnerable options with correlated credit risk under jump-diffusion processes. Journal of Futures Markets, 2014, 34(10): 957–979.

[16] Xu W, Xu W, Li H, et al. A jump-diffusion approach to modelling vulnerable option pricing. Finance Research Letters, 2012, 9(1): 48–56.

[17] Ait-Sahalia Y, Cacho-Diaz J, Laeven R J. Modeling financial contagion using mutually exciting jump processes. Journal of Financial Economics, 2015, 117(3): 585–606.

[18] Fulop A, Li J, Yu J. Self-exciting jumps, learning and asset pricing implications. Review of Financial Studies, 2015, 28(3): 876–912.

[19] 陈浪南, 孙坚强. 股票市场资产收益的跳跃行为研究. 经济研究, 2010, 45(4): 54–66.
Chen L N, Sun J Q. The research on jump dynamics in stock market asset returns. Economic Research Journal, 2010, 45(4): 54–66. (in Chinese)

[20] 马 勇, 张卫国, 闫杜娟. 中国股市暴涨暴跌的交互作用及其预测. 系统管理学报, 2014, 23(5): 755–760.
Ma Y, Zhang W G, Yan D J. The interaction between large rises and large declines in China's stock market and its prediction. Journal of Systems & Management, 2014, 23(5): 755–760. (in Chinese)

[21] Ma Y, Pan D T, Wang T Y. Exchange options under clustered jumps dynamics. Quantitative Finance, 2020, 20(6): 949–967.

[22] Jing B, Li S H, Ma Y. Pricing VIX options with volatility clustering. Journal of Futures Markets, 2020, 40(6): 928–944.

[23] Ma Y, Shrestha K, Xu W D. Pricing vulnerable options with jump clustering. Journal of Futures Markets, 2017, 37(12): 1155–1178.

[24] Schobel R, Zhu J. Stochastic volatility with an Ornstein-Uhlenbeck Process: An extension. European Finance Review, 1999, 3(1): 23–46.

[25] Ma J. Pricing foreign equity options with stochastic correlation and volatility. Annals of Economics and Finance, 2009, 10(2): 303–327.

[26] Branger N, Muck M. Keep on smiling: The pricing of quanto options when all covariances are stochastic. Journal of Banking and Finance, 2012, 36(6): 1577–1591.

[27] Teng L, Ehrhardt M, Günther M. Quanto pricing in stochastic correlation models. International Journal of Theoretical and Applied Finance, 2018, 21(5): 1850038.

[28] Teng L, Ehrhardt M, Günther M. On the Heston model with stochastic correlation. International Journal of Theoretical and Applied Finance, 2016, 19(6): 1–25.

[29] Duffie D, Pan J, Singleton K. Transform analysis and asset pricing for affine jump-diffusions. Econometrica, 2000, 68(6): 1343–1376.

[30] Grzelak L, Oosterlee C. On the Heston model with stochastic interest rates. SIAM Journal on Financial Mathematics, 2011, 2(1): 255–286.

[31] Carr P P, Madan D. Option valuation using the fast Fourier transform. Journal of Computational Finance, 1999, 2(4): 61–73.

[32] Heston S L. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial, 1993, 6(2): 327–343.

[33] Dempster M A H, Hong S S G. Spread option valuation and the fast Fourier transform // Mathematical Finance–Bachelier Congress 2000. Berlin: Springer, 2002: 203–220.

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附录 A(定理 1 的证明)

运用 Feynman-Kac 定理可知 X^{C} 和 Y^{C} 的联合特征函数 $f_{\mathrm{C}}(\phi_1,\phi_2,\tau)$满足如下偏微分方程

$$
\begin{aligned}
&-\frac{\partial f_{\mathrm{C}}}{\partial \tau}+(r-\frac{1}{2}\nu)\frac{\partial f_{\mathrm{C}}}{\partial X^{\mathrm{C}}}+(r-\frac{1}{2}\sigma^2)\frac{\partial f_{\mathrm{C}}}{\partial Y^{\mathrm{C}}}+\beta_\nu(\theta_\nu-\nu)\frac{\partial f_{\mathrm{C}}}{\partial \nu}+(\beta_{\mathrm{r}}(\theta_{\mathrm{r}}-r)-\sigma_{\mathrm{r}}^2L)\frac{\partial f_{\mathrm{C}}}{\partial r}+\\
&\kappa_\rho(\theta_\rho-\rho)\frac{\partial f_{\mathrm{C}}}{\partial \rho}+\frac{1}{2}\nu\frac{\partial^2 f_{\mathrm{C}}}{\partial (X^{\mathrm{C}})^2}+\frac{1}{2}\sigma^2\frac{\partial^2 f_{\mathrm{C}}}{\partial (Y^{\mathrm{C}})^2}+\frac{1}{2}\sigma_\nu^2\nu\frac{\partial^2 f_{\mathrm{C}}}{\partial \nu^2}+\frac{1}{2}\sigma_{\mathrm{r}}^2\frac{\partial^2 f_{\mathrm{C}}}{\partial r^2}+\\
&\frac{1}{2}\eta_\rho^2\frac{\partial^2 f_{\mathrm{C}}}{\partial \rho^2}+\rho_1\sigma_\nu\nu\frac{\partial^2 f_{\mathrm{C}}}{\partial X^{\mathrm{C}}\partial \nu}+\sigma\rho\sqrt{\nu}\frac{\partial^2 f_{\mathrm{C}}}{\partial X^{\mathrm{C}}\partial Y^{\mathrm{C}}}+\rho_1\sigma\sigma_\nu\rho\sqrt{\nu}\frac{\partial^2 f_{\mathrm{C}}}{\partial Y^{\mathrm{C}}\partial \nu}=0.
\end{aligned}\tag{A.1}
$$

方程(A.1)的边界条件为 $f_{\mathrm{C}}(\phi_1,\phi_2,0)=\mathrm{e}^{i\phi_1X^{\mathrm{C}}+i\phi_2Y^{\mathrm{C}}}$. 而方程(A.1)是非线性的偏微分方程, 仿射过程的一般结果在这里不再适用, 这时需要对其非仿射项进行合理近似以便产生仿射形式. 从方程(A.1) 可知, 其非仿射项是 $\sigma\rho(t)\sqrt{\nu(t)}$, $\rho_1\sigma\sigma_\nu\rho(t)\sqrt{\nu(t)}$. 因此, 只需对 $\rho(t)\sqrt{\nu(t)}$ 进行合理近似. 根据文献[28] 可得近似结果

$$\rho(t)\sqrt{\nu(t)}\approx\rho(t)\mathrm{E}^{Q^{\mathrm{T}}}\left[\sqrt{\nu(t)}\right].\tag{A.2}$$

根据文献[30],

$$\mathrm{E}^{Q^{\mathrm{T}}}\left[\sqrt{\nu(t)}\right]\approx m+n\mathrm{e}^{-lt},\tag{A.3}$$

其中 $\hat{d}=\sqrt{\left(\nu(0)\mathrm{e}^{-\beta_\nu}-\frac{\sigma_\nu^2\left(1-\mathrm{e}^{-\beta_\nu}\right)}{4\beta_\nu}\right)+\theta_\nu\left(1-\mathrm{e}^{-\beta_\nu}\right)+\frac{\sigma_\nu^2\theta_\nu\left(1-\mathrm{e}^{-\beta_\nu}\right)^2}{8\beta_\nu\theta_\nu+8\beta_\nu\mathrm{e}^{-\beta_\nu}(\nu(0)-\theta_\nu)}}$, $m=\sqrt{\theta_\nu-\frac{\sigma_\nu^2}{8\beta_\nu}}$, $n=\sqrt{\nu(0)}-m$, $l=-\ln\left(n^{-1}(\hat{d}-m)\right)$.

根据文献[29], 首先假设式(A.1)中的特征函数 $f_{\mathrm{C}}(\phi_1,\phi_2,\tau)$具有如下形式

$$f_{\mathrm{C}}(\phi_1,\phi_2,\tau)=\exp\left(A(\phi_1,\phi_2,\tau)\nu+B(\phi_1,\phi_2,\tau)r+C(\phi_1,\phi_2,\tau)\rho+D(\phi_1,\phi_2,\tau)+i\phi_1X^{\mathrm{C}}+i\phi_2Y^{\mathrm{C}}\right),\tag{A.4}$$

其中边界条件$A(\phi_1,\phi_2,0)=0, B(\phi_1,\phi_2,0)=0, C(\phi_1,\phi_2,0)=0, D(\phi_1,\phi_2,0)=0$. 将式(A.4) 代入式(A.1)以及将式(A.2), 式(A.3)代入式(A.1)近似替换, 得到如下常微分方程组

$$\frac{\mathrm{d}A}{\mathrm{d}\tau}=\frac{1}{2}\sigma_\nu^2A^2+(\rho_1\sigma_\nu i\phi_1-\beta_\nu)A-\frac{1}{2}(i\phi_1+\phi_1^2),\tag{A.5}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\tau}=-\beta_{\mathrm{r}}B+i(\phi_1+\phi_2),\tag{A.6}$$

$$\frac{\mathrm{d}C}{\mathrm{d}\tau}=-\kappa_\rho C+\rho_1\sigma\sigma_\nu i\phi_2\mathrm{E}^{Q^{\mathrm{T}}}\left[\sqrt{\nu(t)}\right]A-\sigma\phi_1\phi_2\mathrm{E}^{Q^{\mathrm{T}}}\left[\sqrt{\nu(t)}\right],\tag{A.7}$$

$$\frac{\mathrm{d}D}{\mathrm{d}\tau}=\beta_\nu\theta_\nu A+\kappa_\rho\theta_\rho C+(\beta_{\mathrm{r}}\theta_{\mathrm{r}}-\sigma_{\mathrm{r}}^2L)B+\frac{1}{2}\sigma_{\mathrm{r}}^2B^2+\frac{1}{2}\eta_\rho^2C^2-\frac{1}{2}\sigma^2\phi_2(\phi_2+i),\tag{A.8}$$

其中式(A.5)是关于A 的常系数 Riccati 方程, 式(A.6)是关于 B 的一元线性常微分方程. 通过简单计算可直接求得

$$A(\phi_1,\phi_2,\tau)=\frac{\tilde{d}(\phi_1)+d(\phi_1)}{\sigma_\nu^2}\cdot\frac{1-\mathrm{e}^{-d(\phi_1)\tau}}{1-g(\phi_1)\mathrm{e}^{-d(\phi_1)\tau}},\tag{A.9}$$

$$B(\phi_1,\phi_2,\tau)=\frac{i(\phi_1+\phi_2)(1-\mathrm{e}^{-\beta_\mathrm{r}\tau})}{\beta_\mathrm{r}}, \tag{A.10}$$

其中 $d(\phi_1)=\sqrt{(\rho_1\sigma_\nu i\phi_1-\beta_\nu)^2+\sigma_\nu^2(i\phi_1+\phi_1^2)}, g(\phi_1)=\frac{\tilde{d}(\phi_1)+d(\phi_1)}{\tilde{d}(\phi_1)-d(\phi_1)}, \tilde{d}(\phi_1)=\rho_1\sigma_\nu i\phi_1-\beta_\nu$.

对于 $C(\phi_1,\phi_2,\tau)$的求解, 首先进行如下近似

$$1-\mathrm{e}^{-l_1\tau}\approx\frac{1-\mathrm{e}^{-d(\phi_1)\tau}}{1-g(\phi_1)\mathrm{e}^{-d(\phi_1)\tau}}, \tag{A.11}$$

其中 $l_1=-\ln(\frac{\mathrm{e}^{-d(\phi_1)}-g(\phi_1)\mathrm{e}^{-d(\phi_1)}}{1-g(\phi_1)\mathrm{e}^{-d(\phi_1)}})$. 式(A.11)的详细证明见文献[28], 于是

$$A(\phi_1,\phi_2,\tau)=C_1(1-\mathrm{e}^{-l_1\tau}), \tag{A.12}$$

其中 $C_1=\frac{\tilde{d}(\phi_1)+d(\phi_1)}{\sigma_\nu^2}$. 将$(A.3)$, $(A.12)$代入$(A.7)$可得

$$\frac{\mathrm{d}C}{\mathrm{d}\tau}=-\kappa_\rho C+\rho_1\sigma\sigma_\nu i\phi_2C_1(m+n\mathrm{e}^{-l(T-\tau)})(1-\mathrm{e}^{-l_1\tau})-\sigma\phi_1\phi_2(m+n\mathrm{e}^{-l(T-\tau)}), \tag{A.13}$$

从而

$$\begin{aligned}C(\phi_1,\phi_2,\tau)=&\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)m}{\kappa_\rho}+\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)n}{l+\kappa_\rho}\mathrm{e}^{l(\tau-T)}-\\&\frac{\rho_1\sigma\sigma_\nu i\phi_2C_1m}{\kappa_\rho-l_1}\mathrm{e}^{-l_1\tau}-\frac{\rho_1\sigma\sigma_\nu i\phi_2C_1n}{\kappa_\rho+l-l_1}\mathrm{e}^{-lT+(l-l_1)\tau}+C_2\mathrm{e}^{-\kappa_\rho\tau},\end{aligned} \tag{A.14}$$

其中

$$C_2=-\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)m}{\kappa_\rho}-\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)n}{l+\kappa_\rho}\mathrm{e}^{-lT}+\frac{\rho_1\sigma\sigma_\nu i\phi_2C_1m}{\kappa_\rho-l_1}+\frac{\rho_1\sigma\sigma_\nu i\phi_2C_1n}{\kappa_\rho+l-l_1}\mathrm{e}^{-lT}.$$

将式(A.9), 式(A.10), 式(A.14)和 $L(t,T)=\frac{1}{\beta_\mathrm{r}}(1-\mathrm{e}^{-\beta_\mathrm{r}(T-t)})$代入(A.8)直接积分得

$$\begin{aligned}D(\phi_1,\phi_2,\tau)=&\beta_\nu\theta_\nu H_0(\phi_1,\tau)+\kappa_\rho\theta_\rho H_1(\phi_1,\phi_2,\tau)+\beta_\mathrm{r}\theta_\mathrm{r}H_2(\phi_1,\phi_2,\tau)-\sigma_\mathrm{r}^2H_3(\phi_1,\phi_2,\tau)+\\&\frac{1}{2}\sigma_\mathrm{r}^2H_4(\phi_1,\phi_2,\tau)+\frac{1}{2}\eta_\rho^2H_5(\phi_1,\phi_2,\tau)-\frac{1}{2}\sigma^2\phi_2(\phi_2+i)\tau,\end{aligned} \tag{A.15}$$

其中

$$\begin{aligned}H_0(\phi_1,\tau)=&-\frac{2}{\sigma_\nu^2}((\frac{\tilde{d}(\phi_1)+d(\phi_1)}{2})\tau+\log(\frac{-\tilde{d}(\phi_1)+d(\phi_1)+(\tilde{d}(\phi_1)+d(\phi_1))\mathrm{e}^{-d(\phi_1)\tau}}{2d(\phi_1)})),\\H_1(\phi_1,\phi_2,\tau)=&\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)m}{\kappa_\rho}\tau+\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)n}{(l+\kappa_\rho)l}\mathrm{e}^{l(\tau-T)}-\frac{C_2}{\kappa_\rho}\mathrm{e}^{-\kappa_\rho\tau}+\\&\frac{\rho_1\sigma\sigma_\nu i\phi_2C_1m}{(\kappa_\rho-l_1)l_1}\mathrm{e}^{-l_1\tau}-\frac{\rho_1\sigma\sigma_\nu i\phi_2C_1n}{(\kappa_\rho+l-l_1)(l-l_1)}\mathrm{e}^{-lT+(l-l_1)\tau}-H_{1c}(\phi_1,\phi_2),\\H_2(\phi_1,\phi_2,\tau)=&\frac{i(\phi_1+\phi_2)}{\beta_\mathrm{r}^2}\mathrm{e}^{-\beta_\mathrm{r}\tau}+\frac{i(\phi_1+\phi_2)}{\beta_\mathrm{r}}\tau-\frac{i(\phi_1+\phi_2)}{\beta_\mathrm{r}^2},\\H_3(\phi_1,\phi_2,\tau)=&\frac{i(\phi_1+\phi_2)}{2\beta_\mathrm{r}^3}(1-\mathrm{e}^{-2\beta_\mathrm{r}\tau})+\frac{2i(\phi_1+\phi_2)}{\beta_\mathrm{r}^3}(\mathrm{e}^{-\beta_\mathrm{r}\tau}-1)+\frac{i(\phi_1+\phi_2)}{\beta_\mathrm{r}^2}\tau,\\H_4(\phi_1,\phi_2,\tau)=&\frac{(\phi_1+\phi_2)^2}{2\beta_\mathrm{r}^3}(\mathrm{e}^{-2\beta_\mathrm{r}\tau}-1)-\frac{2(\phi_1+\phi_2)^2}{\beta_\mathrm{r}^3}(\mathrm{e}^{-\beta_\mathrm{r}\tau}-1)-\frac{(\phi_1+\phi_2)^2}{\beta_\mathrm{r}^2}\tau,\\H_5(\phi_1,\phi_2,\tau)=&I_1\tau+I_2\mathrm{e}^{2l(\tau-T)}+I_3\mathrm{e}^{-2l_1\tau}+I_4\mathrm{e}^{2(l-l_1)\tau-2lT}+I_5\mathrm{e}^{l(\tau-T)}+I_6\mathrm{e}^{-l_1\tau}+\\&(I_7+I_8)\mathrm{e}^{(l-l_1)\tau-lT}+I_9\mathrm{e}^{(2l-l_1)\tau-2lT}+I_{10}\mathrm{e}^{(l-2l_1)\tau-lT}+I_{11}\mathrm{e}^{-\kappa_\rho\tau}+\\&I_{12}\mathrm{e}^{(l-\kappa_\rho)\tau-lT}+I_{13}\mathrm{e}^{-(l_1+\kappa_\rho)\tau}+I_{14}\mathrm{e}^{(l-l_1-\kappa_\rho)\tau-lT}-\frac{1}{2\kappa_\rho}C_2^2\mathrm{e}^{-2\kappa_\rho\tau}-H_{5c}(\phi_1,\phi_2),\\H_{1c}(\phi_1,\phi_2)=&\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)n}{(l+\kappa_\rho)l}\mathrm{e}^{-lT}+\frac{\rho_1\sigma\sigma_\nu i\phi_2C_1m}{(\kappa_\rho-l_1)l_1}-\frac{\rho_1\sigma\sigma_\nu i\phi_2C_1n}{(\kappa_\rho+l-l_1)(l-l_1)}\mathrm{e}^{-lT}-\frac{C_2}{\kappa_\rho},\\H_{5c}(\phi_1,\phi_2)=&I_2\mathrm{e}^{-2lT}+I_3+I_4\mathrm{e}^{-2lT}+I_5\mathrm{e}^{-lT}+I_6+(I_7+I_8)\mathrm{e}^{-lT}+I_9\mathrm{e}^{-2lT}+I_{10}\mathrm{e}^{-lT}+I_{11}+\\&I_{12}\mathrm{e}^{-lT}+I_{13}+I_{14}\mathrm{e}^{-lT}-\frac{1}{2\kappa_\rho}C_2^2,\end{aligned}$$

$$I_1=\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)^2m^2}{\kappa_\rho^2},\quad I_2=\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)^2n^2}{2l(l+\kappa_\rho)^2},\quad I_3=-\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1m)^2}{2l_1(\kappa_\rho-l_1)^2},$$
$$I_4=\frac{(\rho_1\sigma\sigma_\nu i\phi_2C_1n)^2}{2(l-l_1)(\kappa_\rho+l-l_1)^2},\quad I_5=\frac{2(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)nm}{\kappa_\rho l(l+\kappa_\rho)},$$
$$I_6=\frac{2(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)(\rho_1\sigma\sigma_\nu i\phi_2C_1)m^2}{\kappa_\rho l_1(\kappa_\rho-l_1)},\quad I_7=-\frac{2(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)\rho_1\sigma\sigma_\nu i\phi_2C_1mn}{\kappa_\rho(l-l_1)(\kappa_\rho+l-l_1)},$$
$$I_8=-\frac{2(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)(\rho_1\sigma\sigma_\nu i\phi_2C_1)mn}{(l+\kappa_\rho)(\kappa_\rho-l_1)(l-l_1)},\quad I_9=-\frac{2(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)(\rho_1\sigma\sigma_\nu i\phi_2C_1)n^2}{(2l-l_1)(l+\kappa_\rho)(\kappa_\rho+l-l_1)},$$
$$I_{10}=\frac{2(\rho_1\sigma\sigma_\nu i\phi_2C_1)(\rho_1\sigma\sigma_\nu i\phi_2C_1)mn}{(\kappa_\rho-l_1)(\kappa_\rho+l-l_1)(l-2l_1)},\quad I_{11}=-\frac{2(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)C_2m}{\kappa_\rho^2},$$
$$I_{12}=\frac{2(\rho_1\sigma\sigma_\nu i\phi_2C_1-\sigma\phi_1\phi_2)C_2n}{(l+\kappa_\rho)(l-\kappa_\rho)},\quad I_{13}=\frac{2\rho_1\sigma\sigma_\nu i\phi_2C_1C_2m}{(\kappa_\rho-l_1)(\kappa_\rho+l_1)},\quad I_{14}=-\frac{2\rho_1\sigma\sigma_\nu i\phi_2C_1C_2n}{(\kappa_\rho+l-l_1)(l-l_1-\kappa_\rho)}.$$

最终求得 $A(\phi_1,\phi_2,\tau),B(\phi_1,\phi_2,\tau),C(\phi_1,\phi_2,\tau)$和$D(\phi_1,\phi_2,\tau)$, 进而求得 $f_{\rm C}(\phi_1,\phi_2,\tau)$. 通过式(A.4)和式(15)可得 $X(t)$和$Y(t)$的联合特征函数

$$f(\phi_1,\phi_2,\tau)=f_{\rm C}(\phi_1,\phi_2,\tau)f_{\rm J}(\phi_1,\phi_2,\tau).\tag{A.16}$$

附录 B(定理 2 的证明)

令

$$C_1(k,\zeta)={\rm E}^{Q^{\rm T}}\left[({\rm e}^{X(T)}-{\rm e}^k)^+1_{\{{\rm e}^{Y(T)}\geqslant{\rm e}^\zeta\}}\mid\mathcal{F}(t)\right],\tag{B.1}$$

$$C_2(k,\zeta)={\rm E}^{Q^{\rm T}}\left[{\rm e}^{Y(T)}({\rm e}^{X(T)}-{\rm e}^k)^+1_{\{{\rm e}^{Y(T)}<{\rm e}^\zeta\}}\mid\mathcal{F}(t)\right],\tag{B.2}$$

则式(19)可变为

$$C(k,\zeta)=P(t,T)C_1(k,\zeta)+P(t,T)\frac{(1-\omega)}{D}C_2(k,\zeta).\tag{B.3}$$

下面通过傅里叶变换求解 $C_1(k,\zeta)$和 $C_2(k,\zeta)$的表达式, 进而求得$(B.3)$中期权价格的表达式. 因为 C_1 和 C_2 不是平方可积的, 因此不能直接使用傅里叶变换. 根据文献[33], 需要对 $C_1(k,\zeta)$和 $C_2(k,\zeta)$进行修正, 令

$$c_1(k,\zeta)={\rm e}^{\alpha_1k}{\rm e}^{\beta_1\zeta}C_1(k,\zeta),\ \ c_2(k,\zeta)={\rm e}^{\alpha_1k}{\rm e}^{-\beta_2\zeta}C_2(k,\zeta),\tag{B.4}$$

其中 $\alpha_1,\beta_1,\beta_2>0$ 为阻尼系数, 此时 $c_1(k,\zeta)$和 $c_2(k,\zeta)$是平方可积的, 其傅里叶变换分别为

$$\hat{c}_1(\delta_1,\delta_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}{\rm e}^{i\delta_1k}{\rm e}^{i\delta_2\zeta}c_1(k,\zeta){\rm d}k{\rm d}\zeta,\tag{B.5}$$

$$\hat{c}_2(\delta_1,\delta_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}{\rm e}^{i\delta_1k}{\rm e}^{i\delta_2\zeta}c_2(k,\zeta){\rm d}k{\rm d}\zeta.\tag{B.6}$$

对 $\hat{c}_1(\delta_1,\delta_2)$进行计算并运用 Fubini 定理可得

$$\begin{aligned}\hat{c}_1(\delta_1,\delta_2)&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}{\rm e}^{(i\delta_1+\alpha_1)k}{\rm e}^{(i\delta_2+\beta_1)\zeta}{\rm E}^{Q^{\rm T}}\left[({\rm e}^{X(T)}-{\rm e}^k)^+1_{\{{\rm e}^{Y(T)}\geqslant{\rm e}^\zeta\}}\mid\mathcal{F}(t)\right]{\rm d}k{\rm d}\zeta\\&={\rm E}^{Q^{\rm T}}\left[\int_{-\infty}^{+\infty}{\rm e}^{(i\delta_2+\beta_1)\zeta}1_{\{{\rm e}^{Y(T)}\geqslant{\rm e}^\zeta\}}\int_{-\infty}^{X(T)}({\rm e}^{(i\delta_1+\alpha_1)k}{\rm e}^{X(T)}-{\rm e}^{(i\delta_1+\alpha_1+1)k}){\rm d}k{\rm d}\zeta\mid\mathcal{F}(t)\right]\\&=\frac{1}{\alpha_1^2+\alpha_1-\delta_1^2+i(2\alpha_1+1)\delta_1}{\rm E}^{Q^{\rm T}}\left[\int_{-\infty}^{Y(T)}{\rm e}^{(i\delta_2+\beta_1)\zeta}{\rm e}^{(i\delta_1+\alpha_1+1)X(T)}{\rm d}\zeta\mid\mathcal{F}(t)\right]\\&=\frac{1}{(\alpha_1^2+\alpha_1-\delta_1^2+i(2\alpha_1+1)\delta_1)(i\delta_2+\beta_1)}{\rm E}^{Q^{\rm T}}\left[{\rm e}^{(i\delta_1+\alpha_1+1)X(T)+(i\delta_2+\beta_1)Y(T)}\mid\mathcal{F}(t)\right]\\&=\frac{f(\delta_1-(\alpha_1+1)i,\delta_2-\beta_1i,\tau)}{(\alpha_1^2+\alpha_1-\delta_1^2+i(2\alpha_1+1)\delta_1)(i\delta_2+\beta_1)}.\end{aligned}$$

类似可证

$$\hat{c}_2(\delta_1,\delta_2)=\frac{f(\delta_1-(\alpha_1+1)i,\delta_2-(1-\beta_2)i,\tau)}{(\alpha_1^2+\alpha_1-\delta_1^2+i(2\alpha_1+1)\delta_1)(-i\delta_2+\beta_2)}.$$

对 $\hat{c}_1(\delta_1,\delta_2)$和 $\hat{c}_2(\delta_1,\delta_2)$进行傅里叶逆变换可得

$$C_1(k,\zeta)=\frac{{\rm e}^{-\alpha_1k-\beta_1\zeta}}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}{\rm e}^{-i(\delta_1k+\delta_2\zeta)}\hat{c}_1(\delta_1,\delta_2){\rm d}\delta_1{\rm d}\delta_2,$$

$$C_2(k,\zeta)=\frac{\mathrm{e}^{-\alpha_1 k+\beta_2\zeta}}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i(\delta_1 k+\delta_2\zeta)}\hat{c}_2(\delta_1,\delta_2)\mathrm{d}\delta_1\mathrm{d}\delta_2.$$

最终, 可得脆弱欧式看涨期权价格

$$C(k,\zeta)=P(t,T)C_1(k,\zeta)+P(t,T)\tfrac{(1-\omega)}{D}C_2(k,\zeta).$$

附录 C(期权定价的 FFT 算法过程)

根据文献[33], 可以使用梯形法则计算期权价值表达式(20)中的双重积分. 首先, 对式(21)中涉及 $C_1(k,\zeta)$的傅里叶积分进行如下和式近似表示

$$C_1(k,\zeta)\approx\frac{\mathrm{e}^{-\alpha_1 k-\beta_1\zeta}}{(2\pi)^2}\sum_{a_1=0}^{N-1}\sum_{b_1=0}^{N-1}\mathrm{e}^{-i(s_{a1}k+s_{b1}\zeta)}\hat{c}_1(s_{a1},s_{b1})\omega_1\omega_2, \tag{C.1}$$

其中 $s_{a1}=(a_1-\frac{N}{2})\omega_1, s_{b1}=(b_1-\frac{N}{2})\omega_2$ 和 $a_1,b_1=0,1,\ldots,N-1$. N 是区间的数目, ω_1 和 ω_2 都表示步长.

在二维快速傅里叶变换算法中, 定义

$$k_{p_1}=(p_1-\frac{N}{2})\tilde{\lambda}_1,\ \ \zeta_{q_1}=(q_1-\frac{N}{2})\tilde{\lambda}_2,$$

其中 $\tilde{\lambda}_1\omega_1=\tilde{\lambda}_2\omega_2=\frac{2\pi}{N}, 0\leqslant p_1,q_1\leqslant N$.

对 $N\times N$ 个不同执行价格对数和不同违约边界对数, $C_1(k_{p_1},\zeta_{q_1})$可近似表示为[33]

$$C_1(k_{p_1},\zeta_{q_1})\approx\frac{\mathrm{e}^{-\alpha_1 k_{p_1}-\beta_1\zeta_{q_1}}}{(2\pi)^2}\varphi_1(t,T)\omega_1\omega_2, \tag{C.2}$$

其中 $\varphi_1(t,T)=(-1)^{p_1+q_1}\sum\limits_{a_1=0}^{N-1}\sum\limits_{b_1=0}^{N-1}\mathrm{e}^{-\frac{2\pi i}{N}(a_1p_1+b_1q_1)}[(-1)^{a_1+b_1}\hat{c}_1(s_{a1},s_{b1})]$.

类似, 对 $C_2(k,\zeta)$的傅里叶积分进行如下和式近似

$$C_2(k,\zeta)\approx\frac{\mathrm{e}^{-\alpha_1 k+\beta_2\zeta}}{(2\pi)^2}\sum_{a_2=0}^{N-1}\sum_{b_2=0}^{N-1}\mathrm{e}^{-i(s_{a2}k+s_{b2}\zeta)}\hat{c}_2(s_{a2},s_{b2})\omega_1\omega_2, \tag{C.3}$$

其中 $s_{a2}=(a_2-\frac{N}{2})\omega_1, s_{b2}=(b_2-\frac{N}{2})\omega_2, a_2,b_2=0,\ldots,N-1$.

同样, 对于不同执行价格对数和不同违约边界对数的 $C_2(k_{p_1},\zeta_{q_1})$ 有如下数值近似[33]

$$C_2(k_{p_1},\zeta_{q_1})\approx\frac{\mathrm{e}^{-\alpha_1 k_{p_1}+\beta_2\zeta_{q_1}}}{(2\pi)^2}\varphi_2(t,T)\omega_1\omega_2, \tag{C.4}$$

其中 $\varphi_2(t,T)=(-1)^{p_1+q_1}\sum\limits_{a_2=0}^{N-1}\sum\limits_{b_2=0}^{N-1}\mathrm{e}^{-\frac{2\pi i}{N}(a_2p_1+b_2q_1)}[(-1)^{a_2+b_2}\hat{c}_2(s_{a2},s_{b2})]$.

最后, 关于 $N\times N$ 个不同执行价格对数和违约边界对数的脆弱期权价格的近似值为

$$\begin{aligned}C(k_{p_1},\zeta_{q_1})=&P(t,T)C_1(k_{p_1},\zeta_{q_1})+P(t,T)\frac{(1-\omega)}{D}C_2(k_{p_1},\zeta_{q_1})\\ \approx&P(t,T)\frac{\mathrm{e}^{-\alpha_1 k_{p_1}-\beta_1\zeta_{q_1}}}{(2\pi)^2}\varphi_1(t,T)\omega_1\omega_2+P(t,T)\frac{(1-\omega)}{D}\frac{\mathrm{e}^{-\alpha_1 k_{p_1}+\beta_2\zeta_{q_1}}}{(2\pi)^2}\varphi_2(t,T)\omega_1\omega_2,\end{aligned}$$

此时可以利用 FFT 对以上和式进行计算.